Renormalization of vacuum energy in linearized quantum gravity

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Abstract

In linearized quantum gravity, a shift of the average energy-momentum can be compensated by a shift of the average gravitational field. This allows a renormalization scheme that naturally removes the contribution of quantum vacuum fluctuations to the cosmological constant, solving the old cosmological-constant problem for weak gravitational fields.

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As is well known, the average energy of a quantum field diverges in any quantum state, including the vacuum. In non-gravitational physics the zero-energy point can be chosen arbitrarily, so energy can be renormalized by a simple shift of energy chosen such that the energy of the vacuum is equal to zero. However, in gravitational physics the energy is the source of the gravitational field, so the zero-energy point is not arbitrary. Consequently, the quantum vacuum energy contributes to the cosmological constant, by a contribution 120 orders of magnitude larger than the measured one. This problem is known as the cosmological-constant problem [1, 2, 3, 4, 5]. In the old formulation of the problem [1, 2] one would like to find a theoretical mechanism that makes this vacuum contribution to the cosmological constant vanishing. In the new, more ambitious, formulation of the problem [3, 4, 5] one would like to explain why the sum of all possible contributions to the cosmological constant, including that of the vacuum energy, is of the same order of magnitude as the matter density of the universe.

The cosmological-constant problem usually appears at the level of semi-classical gravity, in which matter is quantized but gravity is not. It is very likely that the

solution of the problem requires a fully quantized gravity. A renormalization of the energy-momentum could be compensated by a renormalization of the gravitational field, which could make the subtraction of the vacuum energy-momentum consistent. Unfortunately, a fully quantized gravity is too difficult to deal with, so one is forced to use some approximations. In this letter we deal with linearized quantum gravity and show how in this context the old cosmological-constant problem can be solved in a very simple way. It also represents a very generic solution of the problem, in the sense that it does not depend on technical details specifying how exactly gravity is quantized.

In an appropriate system of units, the classical Einstein equation can be written as $G_{\mu\nu} = T_{\mu\nu}$. The left-hand side $G_{\mu\nu}$ is the Einstein tensor, which can be expressed as a nonlinear functional of the metric $g_{\mu\nu}$. The right-hand side $T_{\mu\nu}$ is the matter energy-momentum tensor, which can be expressed as a nonlinear functional of the metric $g_{\mu\nu}$ and the matter fields generically denoted by ϕ. To simplify the notation, in the rest of the discussion we suppress the spacetime indices μ, ν. This allows us to write the Einstein equation in a generic form as

$$G[g] = T[g, \phi]. \tag{1}$$

The metric can be written as

$$g = \eta + h, \tag{2}$$

where η is the flat Minkowski metric and h is the dynamical gravitational field. Here η is treated as a fixed metric, so (1) can be written as

$$G[h] = T[h, \phi], \tag{3}$$

where now G and T are some new functionals highly nonlinear in h (see, e.g., [6]).

In quantum gravity, the fields h and ϕ are promoted to the operators $\hat{h}$ and $\hat{\phi}$. Thus, with an appropriate operator ordering, the quantum fields in the Heisenberg picture satisfy the operator Einstein equation

$$G[\hat{h}] = T[\hat{h}, \hat{\phi}]. \tag{4}$$

Hence, for any *physical* quantum state $|\psi\rangle$ we have

$$\langle\psi|G[\hat{h}]|\psi\rangle = \langle\psi|T[\hat{h}, \hat{\phi}]|\psi\rangle. \tag{5}$$

In particular, we choose one special fiducial state $|\Omega\rangle$ (to be specified below), so we have

$$\langle\Omega|G[\hat{h}]|\Omega\rangle = \langle\Omega|T[\hat{h}, \hat{\phi}]|\Omega\rangle. \tag{6}$$

With a generic ordering of operators, the right-hand sides of (5) and (6) are divergent, and hence so are the left-hand sides. Still, we can subtract these two equations, leading to

$$\langle\psi|G[\hat{h}]|\psi\rangle - \langle\Omega|G[\hat{h}]|\Omega\rangle = T^{(\psi)}, \tag{7}$$

where

$$T^{(\psi)} \equiv \langle\psi|T[\hat{h}, \hat{\phi}]|\psi\rangle - \langle\Omega|T[\hat{h}, \hat{\phi}]|\Omega\rangle. \tag{8}$$

We see that (8) looks just as the average energy-momentum tensor in the state $|\psi\rangle$, renormalized by subtraction of the average energy-momentum tensor in the fiducial state $|\Omega\rangle$. Thus, if $|\Omega\rangle$ was the matter vacuum, the right-hand side of (7) would look just like the right-hand side of the semiclassical Einstein-equation with a shifted energy-momentum, such that the vacuum energy-momentum vanishes. If the left-hand side of (7) could be interpreted as the left-hand side of the semiclassical Einstein equation, this would represent a solution of the old cosmological-constant problem, because the vacuum energy-momentum would be removed by a shift of the zero-energy point, without ignoring the fact that the vacuum energy-momentum is also a source for the gravitational field. Unfortunately, the problem is that the left-hand side of the semiclassical Einstein equation should have a form $G[h]$, implying that the left-hand side of (7) *cannot* be interpreted as the left-hand side of the semiclassical Einstein equation. Thus, it is not yet clear how the simple manipulations above can help in solving the cosmological-constant problem.

The situation significantly improves in linearized gravity. The functional $G[h]$ can be expanded in powers of h as

$$G[h] \simeq Lh + \mathcal{O}(h^2), \tag{9}$$

where L is a linear operator (essentially a second-order derivative operator not depending on h) acting on $h(x)$. Linearized gravity is an approximation in which $\mathcal{O}(h^2)$ in (9) is neglected. It is a good approximation for weak gravitational fields. In the lowest order in h we can also write $T[h,\phi] \simeq T[\phi]$, i.e., calculate the energy-momentum tensor with the flat Minkowski background metric η. Therefore, in the lowest order in $\hat{h}$, (7) can be written as

$$Lh^{(\psi)} = T^{(\psi)}, \tag{10}$$

where

$$T^{(\psi)} = \langle\psi|T[\hat{\phi}]|\psi\rangle - \langle\Omega|T[\hat{\phi}]|\Omega\rangle, \tag{11}$$

$$h^{(\psi)} \equiv \langle\psi|\hat{h}|\psi\rangle - \langle\Omega|\hat{h}|\Omega\rangle. \tag{12}$$

We see that (10) looks exactly as a linearized semiclassical Einstein-equation, in which the "classical" gravitational field $h^{(\psi)}$ is actually the average gravitational field renormalized by subtraction of the average gravitational field in the fiducial state $|\Omega\rangle$.

So far, we have not fixed the fiducial state $|\Omega\rangle$. We fix it by requiring that the vacuum energy-momentum $T^{(0)}$ has the cosmological-constant form. In the lowest order in h, this means

$$T^{(0)} = \Lambda\eta, \tag{13}$$

where Λ is an *unspecified* scalar. Eq. (11) then implies

$$\langle 0_\phi|T[\hat{\phi}]|0_\phi\rangle - \langle\Omega|T[\hat{\phi}]|\Omega\rangle = \Lambda\eta, \tag{14}$$

where $|0_\phi\rangle$ is the matter vacuum. But (14) is only possible if $|\Omega\rangle$ is also proportional to a Lorentz invariant matter state, that is, to the matter vacuum. Hence, assuming that the matter vacuum $|0_\phi\rangle$ is unique, $|\Omega\rangle$ must be of the form

$$|\Omega\rangle = |\Omega_h\rangle \otimes |0_\phi\rangle, \tag{15}$$

where $|\Omega_h\rangle$ is some gravitational state. This implies that $T^{(0)} = 0$, i.e., that the contribution of the quantum fluctuations to the cosmological constant in (13) is

$$\Lambda = 0. \tag{16}$$

We also note that (11) is finite and that, due to (15), it corresponds to the usual normal ordering of matter field operators. This is so owing to the fact that the background metric η is chosen to be the flat Minkowski one. (In η was chosen to be a curved background, then (11) would not necessarily be finite [7].) Thus, *the quantum fluctuations of the vacuum do not contribute to the cosmological constant* in linearized quantum gravity.

Further, our renormalization scheme removes even a contribution from a bare cosmological constant. To see this, assume that the classical action contains a cosmological term proportional to $\int d^4x\sqrt{|g|}\Lambda_B$, where Λ_B is a constant representing the bare cosmological constant. The right-hand side of (1) attains an additional term $\Lambda_B g$. In the lowest order in h this becomes $\Lambda_B\eta$, which in quantum theory does *not* become an operator. Consequently, as $\langle\psi|\psi\rangle = \langle\Omega|\Omega\rangle = 1$, the right-hand sides of (5) and (6) attain the same additional term $\Lambda_B\eta$, which cancel each other after the subtraction in (7).

Our results show that a natural value of the cosmological constant in linearized quantum gravity is zero. This solves the old cosmological-constant problem in linearized quantum gravity. But what about the new cosmological-constant problem? Although our approach does not solve it, our result (16) is not in contradiction with the fact that the measured cosmological constant is not really zero. Namely, although $T^{(0)} = 0$, it does not imply that $T^{(\psi)}$ cannot have a contribution of the form $\Lambda^{(\psi)}(x)\eta$, where $\Lambda^{(\psi)}(x)$ is some dynamical contribution to dark energy, depending on the state $|\psi\rangle$. Indeed, such a contribution, if exists, must depend on the matter content of the universe described by the state $|\psi\rangle$, which could explain why the measured energy density of dark energy is of the same order of magnitude as the measured matter density. Thus, the observed cosmological constant could be explained dynamically by a quantum version of a quintessence model.

Another mechanism that may induce an effective cosmological constant is by allowing the existence of more than one vacuum state, having different energies. The vacuum in the first term in (14) may be $|0_{\phi,2}\rangle$, while the vacuum in (15) may be $|0_{\phi,1}\rangle$. Then the difference on the right-hand side of (14) may be finite. Such a mechanism could be relevant for the early cosmological inflation.

A further contribution to an effective cosmological constant could arise from nonlinear corrections to our linearized quantum gravity. At this level such an idea may seem rather speculative, but results from loop quantum gravity already suggest that the early inflation may be a consequence of strong quantum gravitational fields interacting with generic matter [8].

To conclude, in linearized quantum gravity a shift of the average energy-momentum, Eq. (11), can be compensated by a shift of the average gravitational field, Eq. (12). This leads to a simple resolution of the old cosmological-constant problem for weak gravitational fields. It also gives some hints on a possible resolution of the new cosmological-constant problem as well.

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References

[1] S. Weinberg, Rev. Mod. Phys. 61 (1989) 1.

[2] S. Nobbenhuis, Found. Phys. 36 (2006) 613.

[3] V. Sahni, A. Starobinsky, Int. J. Mod. Phys. D 9 (2000) 373.

[4] S.M. Carroll, Living Rev. Rel. 4 (2001) 1.

[5] T. Padmanabhan, Phys. Rep. 380 (2003) 235.

[6] R.P. Feynman, Feynman Lectures on Gravitation, Addison-Wesley, Mass., 1995.

[7] N.D. Birrell, P.C.W. Davies, Quantum Fields in Curved Space, Cambridge Press, NY, 1982.

[8] M. Bojowald, Phys. Rev. Lett. 89 (2002) 261301.